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"On Fuzzy Arithmetic Operations: Some Properties and Distributive Approximations"

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On Fuzzy Arithmetic Operations: Some Properties and Distributive Approximations

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Abstract

We analyze a decomposition of the fuzzy numbers (or intervals) which seems to be of interest in the study of some properties of fuzzy arithmetic operations and, in particular, in the analysis of fuzziness, of shape-preservation (symmetry) and distributivity of multiplication and division.

By the use of the same decomposition, we suggest an approximation of multiplication and division to reduce the overestimation effect and/or to obtain total-distributivity of multiplication and left-distributivity of division.

Finally, we compare the proposed approximation with the results of standard (α -cuts based) fuzzy mathematics and with other new definitions of fuzzy arithmetic operations that recently appeared in the literature.

1. INTRODUCTION AND BASIC RESULTS

The scientific literature on fuzzy arithmetic operations is rich of several approaches to define fuzzy operations having many desired properties that are not always present in the classical extension principle approach (see [14]) or its approximations (see [13]): shape preservation (e.g. [3], [4], [13]), reduction of the overestimation effect (e.g. [6], [7]), requisite constraints (e.g. [9], [8]), distributivity of multiplication and division (e.g. [1], [10], [11], [12]). These problems are essentially approached by joining representations of fuzzy quantities and fuzzy operations (e.g. [2], [5], [6], [9]).

In this paper, we introduce a decomposition of the fuzzy numbers (or intervals) into three additive components that isolate a "crisp part", a "symmetric fuzzy part" and a "profile of symmetry" each capturing precise properties of fuzzy quantities. The decomposition is used to analyze some properties of fuzzy operations and, particularly, distributivity of multiplication (and division) and symmetry of the results of fuzzy operations.

The decomposition is also used to suggest some approximations of fuzzy operations that reduce the range of fuzziness (with respect to the classical exact operations) and assure distributivity.

1.1. Fuzzy Intervals and Numbers. As a definition of a fuzzy number or interval, we adopt the so called α -cut setting:

Definition 1. A fuzzy number (or interval) u is any pair (u^-, u^+) of functions $u^\pm : [0, 1] \longrightarrow \mathbb{R}$ satisfying the following conditions:

(i) $u^- : \alpha \longrightarrow u^-_\alpha \in \mathbb{R}$ is a bounded monotonic increasing (non decreasing) function

$\forall \alpha \in [0, 1]$;

(ii) $u^+ : \alpha \longrightarrow u_\alpha^+ \in \mathbb{R}$ is a bounded monotonic decreasing (non increasing) function

$\forall \alpha \in [0, 1]$;

(iii) $u_\alpha^- \leq u_\alpha^+ \forall \alpha \in [0, 1]$.

We denote the $(\alpha = 1)$ - cut $[u_1^- < u_1^+]$ by $[\hat{u}^-, \hat{u}^+]$; if $\hat{u}^- < \hat{u}^+$ we have a fuzzy interval and if $\hat{u}^- = \hat{u}^+$ we have a fuzzy number.

We use the notation $[u]_\alpha = [u_\alpha^-, u_\alpha^+]$ to denote explicitly the α - cuts of u .

We will also refer to u^- and u^+ as the *lower* and *upper* branches on u (corresponding to left and right branches in the membership function), respectively.

We denote by $\mathbb{F}_{\mathbb{I}}$ the set of fuzzy intervals and by $\mathbb{F} \subset \mathbb{F}_{\mathbb{I}}$ the set of fuzzy numbers.

If $u_\alpha^- = \hat{u}^-$ and $u_\alpha^+ = \hat{u}^+$, $\forall \alpha$ we have a crisp interval or a crisp number; we denote by $\hat{\mathbb{F}}_{\mathbb{I}}$ and by $\hat{\mathbb{F}}$ the corresponding sets.

If $\hat{u}^- = \hat{u}^+ = 0$ we obtain a 0-fuzzy number and denote the corresponding set by \mathbb{F}_0 .

If $u_\alpha^- + u_\alpha^+ = \hat{u}^+ + \hat{u}^-$, $\forall \alpha$ then the fuzzy interval is called symmetric; we denote by $\mathbb{S}_{\mathbb{I}}$ and by \mathbb{S} the sets of symmetric fuzzy intervals and numbers; $\mathbb{S}_0 = \mathbb{S} \cap \mathbb{F}_0$ will be the set of symmetric 0-fuzzy numbers.

We say that u is positive if $u_\alpha^- > 0$, $\forall \alpha \in [0, 1]$ and that u is negative if $u_\alpha^+ < 0$, $\forall \alpha \in [0, 1]$; the sets of positive and negative fuzzy numbers are denoted by \mathbb{F}_+ and \mathbb{F}_- respectively and their symmetric subsets are denoted by \mathbb{S}_+ and \mathbb{S}_- .

For the standard triangular fuzzy numbers and trapezoidal fuzzy intervals we will use the notations $\langle a, b, c \rangle$ with $a \leq b \leq c$ and $\langle a, b, c, d \rangle$ with $a \leq b \leq c \leq d$; their α -cuts are $[\langle a, b, c \rangle]_\alpha = [a + \alpha(b - a), c - \alpha(c - b)]$ and $[\langle a, b, c, d \rangle]_\alpha = [a + \alpha(b - a), d - \alpha(d - c)]$ respectively.

1.2. Standard fuzzy arithmetic operations. If $u = (u^-, u^+)$ and $v = (v^-, v^+)$ are two given fuzzy intervals, the standard arithmetic operations are defined as follows:

Definition 2. (Standard addition, scalar multiplication, subtraction)

For $\alpha \in [0, 1]$:

$$[u + v]_\alpha = [u_\alpha^- + v_\alpha^-, u_\alpha^+ + v_\alpha^+], \quad (1)$$

$$[ku]_\alpha = [\min \{ku_\alpha^-, ku_\alpha^+\}, \max \{ku_\alpha^-, ku_\alpha^+\}], \quad k \in \mathbb{R}; \quad (2)$$

in particular, if $k = -1$, $[-u]_\alpha = [-u_\alpha^+, -u_\alpha^-]$,

$$[u - v]_\alpha = [u_\alpha^- - v_\alpha^+, u_\alpha^+ - v_\alpha^-] \quad , \quad \alpha \in [0, 1]. \quad (3)$$

Definition 3. (Standard multiplication and division)

For $\alpha \in [0, 1]$, $[uv]_\alpha = [(uv)_\alpha^-, (uv)_\alpha^+]$ with

$$(uv)_\alpha^- = \min \{u_\alpha^- v_\alpha^-, u_\alpha^- v_\alpha^+, u_\alpha^+ v_\alpha^-, u_\alpha^+ v_\alpha^+\} \quad (4)$$

$$(uv)_\alpha^+ = \max \{u_\alpha^- v_\alpha^-, u_\alpha^- v_\alpha^+, u_\alpha^+ v_\alpha^-, u_\alpha^+ v_\alpha^+\}.$$

if $0 \notin [v_0^-, v_0^+]$, $[\frac{u}{v}]_\alpha = [(\frac{u}{v})_\alpha^-, (\frac{u}{v})_\alpha^+]$ with

$$\left(\frac{u}{v}\right)_\alpha^- = \min \left\{ \frac{u_\alpha^-}{v_\alpha^-}, \frac{u_\alpha^-}{v_\alpha^+}, \frac{u_\alpha^+}{v_\alpha^-}, \frac{u_\alpha^+}{v_\alpha^+} \right\} \quad (5)$$

$$\left(\frac{u}{v}\right)_\alpha^+ = \max \left\{ \frac{u_\alpha^-}{v_\alpha^-}, \frac{u_\alpha^-}{v_\alpha^+}, \frac{u_\alpha^+}{v_\alpha^-}, \frac{u_\alpha^+}{v_\alpha^+} \right\}.$$

It is also well known that, in general, distributivity of multiplication and left-distributivity of division are not valid, except for special cases (see [11] or [3]).

2. A DECOMPOSITION OF FUZZY INTERVALS

Consider a generic fuzzy interval u in terms of its α -cuts $[u]_\alpha = [u_\alpha^-, u_\alpha^+]$, $\alpha \in [0, 1]$ and denote the $(\alpha = 1)$ -cut of u by

$$\hat{u} = [\hat{u}^-, \hat{u}^+] \quad (6)$$

so that $\hat{u} \in \hat{\mathbb{F}}_1$; \hat{u} is a crisp number (and u is a fuzzy number) if and only if

$$\hat{u}^- = \hat{u}^+ = \hat{u}.$$

The basic property of the α -cuts is that they are compact intervals and that

$$\alpha' \leq \alpha'' \implies [u]_{\alpha''} \subseteq [u]_{\alpha'}. \quad (7)$$

In particular, it holds:

$$\hat{u} \subseteq [u]_\alpha, \quad \forall \alpha \in [0, 1].$$

The Hukuhara difference of two compact intervals $A = [a_1, a_2]$ and $B = [b_1, b_2]$ is defined by

$$A -_h B = C \iff A = B + C$$

where $B + C = \{b + c \mid b \in B, c \in C\}$ is the usual Minkowski addition. If $B \subseteq A$ then the Hukuhara difference is well defined and given by the compact interval $C = [a_1 - b_1, a_2 - b_2]$, (note that $0 \in C$).

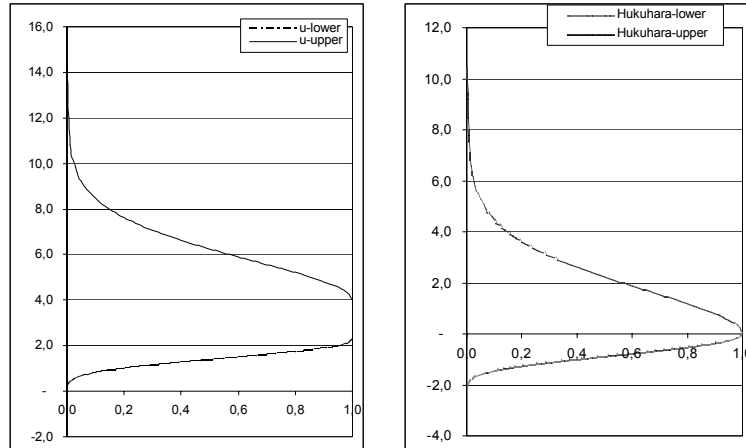
Consider, for $\alpha \in [0, 1]$, the compact intervals:

$$[u]_\alpha^o = u_\alpha -_h \hat{u} = [u_\alpha^- - \hat{u}^-, u_\alpha^+ - \hat{u}^+];$$

it is immediate to verify that the family of sets

$$\left\{ [u]_\alpha^o \mid \alpha \in [0, 1] \right\}$$

defines the α -cuts $[u_\alpha^o, u_\alpha^+]$ of a fuzzy number $\overset{o}{u} \in \mathbb{F}_0$.



General fuzzy interval $u = (u^-, u^+)$ Hukuhara difference $u_\alpha -_h \hat{u}$

In order to capture the asymmetry of u we define the following **profile function** of u :

$$\begin{aligned} \tilde{u} : [0, 1] &\longrightarrow \mathbb{R} \\ \alpha &\longrightarrow \tilde{u}_\alpha \\ \tilde{u}_\alpha &= \frac{\overset{o}{u}_\alpha^- + \overset{o}{u}_\alpha^+}{2} = \frac{u_\alpha^- + u_\alpha^+}{2} - \frac{\hat{u}^- + \hat{u}^+}{2} \end{aligned} \quad (8)$$

It is immediate to verify that:

$$u \in \mathbb{S} \iff \tilde{u}_\alpha = 0 \quad \forall \alpha \in [0, 1]$$

Observe that for $u \in \mathbb{F}_I$,

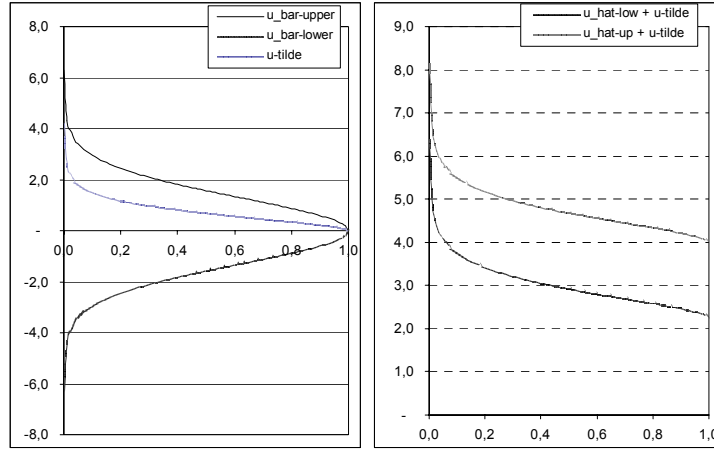
$$\begin{aligned} u_\alpha^+ - \hat{u}^+ - \tilde{u}_\alpha &= \frac{u_\alpha^+ - u_\alpha^-}{2} - \frac{\hat{u}^+ - \hat{u}^-}{2} \\ \hat{u}^- + \tilde{u}_\alpha - u_\alpha^- &= \frac{u_\alpha^+ - u_\alpha^-}{2} - \frac{\hat{u}^+ - \hat{u}^-}{2} \end{aligned}$$

As the last step of our decomposition, we define the \mathbb{S}_0 -component of u as the fuzzy number $\bar{u} \in \mathbb{S}_0$ having α -cuts

$$[-\bar{u}_\alpha, \bar{u}_\alpha],$$

with $\bar{u}_\alpha \geq 0 \quad \forall \alpha$ and \bar{u}_α decreasing, obtained by

$$\bar{u}_\alpha = \frac{u_\alpha^+ - u_\alpha^-}{2} - \frac{\hat{u}^+ - \hat{u}^-}{2}. \quad (9)$$



Fuzzy and profile components Mixed component $(\hat{u}, \tilde{u}, 0)$

So, $\forall \alpha \in [0, 1]$:

$$\begin{cases} u_\alpha^+ = \hat{u}^+ + \tilde{u}_\alpha + \bar{u}_\alpha \\ u_\alpha^- = \hat{u}^- + \tilde{u}_\alpha - \bar{u}_\alpha \end{cases} \quad (10)$$

and we write (10) as

$$\begin{aligned} u &= (\hat{u}, \tilde{u}, \bar{u}) \text{ or} \\ [u]_\alpha &= [\hat{u}^-, \hat{u}^+] + \tilde{u}_\alpha + [-\bar{u}_\alpha, \bar{u}_\alpha] \end{aligned} \quad (11)$$

obtaining a decomposition of the form

$$(fuzzy) = (crisp) + (profile) + (0 - symmetric fuzzy).$$

By use of decomposition (11) we can characterize the fuzzy intervals or numbers as follows:

1. $u \in \mathbb{F}_0 \iff \hat{u}^- = \hat{u}^+ = 0$
2. $u \in \mathbb{F}_- \iff \hat{u}^+ + \tilde{u}_\alpha \leq -\bar{u}_\alpha, \forall \alpha$
3. $u \in \mathbb{F}_+ \iff \hat{u}^- + \tilde{u}_\alpha \geq \bar{u}_\alpha, \forall \alpha$
4. $u \in \mathbb{S} \iff \tilde{u}_\alpha = 0, \forall \alpha$
5. $u \in \mathbb{S}_0 \iff \hat{u}^- = \hat{u}^+ = 0, \tilde{u}_\alpha = 0, \forall \alpha$
6. $u \in \mathbb{S}_- \iff \hat{u}^- \leq -\bar{u}_\alpha, \tilde{u}_\alpha = 0, \forall \alpha$
7. $u \in \mathbb{S}_+ \iff \hat{u}^+ \geq \bar{u}_\alpha, \tilde{u}_\alpha = 0, \forall \alpha$

Note also that $\tilde{u}_1 = 0$ and $-\bar{u}_\alpha \leq \tilde{u}_\alpha \leq \bar{u}_\alpha \forall \alpha \in [0, 1]$ and that the profile function induces a linear operator; in fact, if $u = (\hat{u}, \tilde{u}, \bar{u})$ and $v = (\hat{v}, \tilde{v}, \bar{v})$ then $u + v = (\hat{u} + \hat{v}, \tilde{u} + \tilde{v}, \bar{u} + \bar{v})$ and, $\forall k \in \mathbb{R}$, $ku = (k\hat{u}, k\tilde{u}, k|\bar{u}|)$ and $\widetilde{u + v} = \tilde{u} + \tilde{v}$, $\widetilde{ku} = k\tilde{u}$.

Let's denote by \mathbb{P} the set of all possible profile functions:

$$\begin{aligned} \tilde{u} &: [0, 1] \longrightarrow \mathbb{R} \\ \text{with } \tilde{u}_1 &= 0. \end{aligned}$$

Theorem 4. Any

$$u = (\hat{u}, \tilde{u}, \bar{u}) \in \widehat{\mathbb{F}}_1 \times \mathbb{P} \times \mathbb{S}_0$$

represents a fuzzy interval if and only if the following condition is satisfied by the pair $(\tilde{u}, \bar{u}) \in \mathbb{P} \times \mathbb{S}_0$:

$$\alpha' < \alpha'' \implies |\tilde{u}_{\alpha''} - \tilde{u}_{\alpha'}| \leq \bar{u}_{\alpha'} - \bar{u}_{\alpha''} \quad (12)$$

Proof. If u is a fuzzy interval with \hat{u}, \tilde{u} and \bar{u} defined by (6), (8) and (9) respectively, then (12) is immediate. Suppose now that (12) is valid, then for $\alpha' = \alpha < 1$ and $\alpha'' = 1$ we have

$$|\tilde{u}_\alpha| \leq \bar{u}_\alpha$$

and necessary condition (7) is satisfied. By definition (10) we then have:

$$u_\alpha^- \leq u_\alpha^+ \quad \forall \alpha \text{ (as } \bar{u}_\alpha \geq 0 \text{)}.$$

For $\alpha' < \alpha''$ the following is true

$$u_{\alpha'}^+ = \hat{u}^+ + \tilde{u}_{\alpha'} + \bar{u}_{\alpha'} \geq \hat{u}^+ + \tilde{u}_{\alpha''} + \bar{u}_{\alpha''} = u_{\alpha''}^+$$

and

$$u_{\alpha'}^- = \hat{u}^- + \tilde{u}_{\alpha'} - \bar{u}_{\alpha'} \leq \hat{u}^- + \tilde{u}_{\alpha''} - \bar{u}_{\alpha''} = u_{\alpha''}^-$$

so that u_α^+ is decreasing (not increasing) and u_α^- is increasing (not decreasing). It follows that u is a proper fuzzy interval. ■

Definition 5. (*valid pair* (\tilde{u}, \bar{u}))

We say that a pair $(\tilde{u}, \bar{u}) \in \mathbb{P} \times \mathbb{S}_0$ is a valid pair if it satisfies condition (12), i.e. if $\tilde{u} + \bar{u} \in \mathbb{F}_0$ is a fuzzy number.

In terms of the above definition, we can say that any $u = (\hat{u}, \tilde{u}, \bar{u}) \in \hat{\mathbb{F}}_{\mathbb{I}} \times \mathbb{P} \times \mathbb{S}_0$ represents a fuzzy interval if and only if (\tilde{u}, \bar{u}) is a valid pair. On the other hand, valid pairs of $\mathbb{P} \times \mathbb{S}_0$ are the elements of \mathbb{F}_0 , the 0-fuzzy numbers.

Note that if \tilde{u} and \bar{u} are differentiable functions with respect to $\alpha \in]0, 1[$, then condition (12) can be stated in terms of first derivatives as $|\tilde{u}'_{\alpha}| \leq -\bar{u}'_{\alpha} \quad \forall \alpha \in]0, 1[$. Furthermore, as $\tilde{u}_1 = \bar{u}_1 = 0$ and the absolute local variations of \tilde{u} are not greater than local variations of \bar{u} , it follows that if $(\tilde{u}, \bar{u}) \in \mathbb{P} \times \mathbb{S}_0$ is a valid pair then also $|\tilde{u}_{\alpha}| \leq \bar{u}_{\alpha}$, $\forall \alpha \in [0, 1]$.

Using decomposition $u = (\hat{u}, \tilde{u}, \bar{u})$ with $\hat{u} \in \hat{\mathbb{F}}_{\mathbb{I}}$, $\tilde{u} \in \mathbb{P}$ and $\bar{u} \in \mathbb{S}_0$ and with (\tilde{u}, \bar{u}) a valid pair, the following possibilities are of interest:

$(\hat{u}, 0, 0)$ is crisp; $(0, 0, \bar{u})$ is in \mathbb{S}_0 ; $(\hat{u}, 0, \bar{u})$ is in $\mathbb{S}_{\mathbb{I}}$;
 $(0, \tilde{u}, \bar{u})$ is in \mathbb{F}_0 ; $(\hat{u}, \tilde{u}, \bar{u})$ is in $\mathbb{F}_{\mathbb{I}}$; $(0, \tilde{u}, 0)$ is in \mathbb{P} .

Also a configuration $(\hat{u}, \tilde{u}, 0)$ with $\hat{u} = [\hat{u}^-, \hat{u}^+]$ may be of interest as it represents an interval of fixed length but of varying position, depending on the profile function \tilde{u} : at different degree of possibility α , the position of the fixed length interval changes by following profile \tilde{u} .

If $\varphi_u : \mathbb{R} \rightarrow [0, 1]$ is the membership function of fuzzy interval u , of the form

$$\varphi_u(x) = \begin{cases} 0 & \text{for } x \leq u_0^- \\ u^l(x) & \text{for } x \in [u_0^-, \hat{u}^-] \\ 1 & \text{for } x \in [\hat{u}^-, \hat{u}^+] \\ u^r(x) & \text{for } x \in [\hat{u}^+, u_0^+] \\ 0 & \text{for } x \geq u_0^+ \end{cases}$$

with $u^l(x)$ increasing and $u^r(x)$ decreasing, we have that

$$\forall \alpha \in [0, 1] : \begin{cases} \varphi_u(u_{\alpha}^-) = \alpha = u^l(u_{\alpha}^-) \\ \varphi_u(u_{\alpha}^+) = \alpha = u^r(u_{\alpha}^+) \end{cases}$$

In terms of decomposition (10) we can write the following relations, $\forall \alpha \in]0, 1[$:

$$\begin{cases} u^l(x) = \alpha \iff x = \hat{u}^- + \tilde{u}_{\alpha} - \bar{u}_{\alpha} \\ u^r(x) = \alpha \iff x = \hat{u}^+ + \tilde{u}_{\alpha} + \bar{u}_{\alpha} \end{cases}$$

The fuzziness of u is essentially contained in component $\bar{u} \in \mathbb{S}_0$, while the profile function \tilde{u} is related to the asymmetry of u with respect to crisp component \hat{u} .

Starting with one or more valid pairs, it is not difficult to construct other valid pairs.

For example, if (\tilde{u}, \bar{u}) is given we can consider any function $F : \mathbb{P} \times \mathbb{S}_0 \rightarrow \mathbb{P} \times \mathbb{S}_0$ such that $F(\tilde{u}, \bar{u}) = (\tilde{F}(\tilde{u}, \bar{u}), \bar{F}(\tilde{u}, \bar{u}))$ is a valid pair with $\tilde{F} : \mathbb{P} \times \mathbb{S}_0 \rightarrow \mathbb{P}$ and $\bar{F} : \mathbb{P} \times \mathbb{S}_0 \rightarrow \mathbb{S}_0$. A particular case is related to the extension of a given function $f : \mathbb{R} \rightarrow \mathbb{R}$ to a 0-fuzzy number $u = (0, \tilde{u}, \bar{u}) \in \mathbb{F}_0$ for which

$$\begin{aligned} \widetilde{f(u)} &= \frac{\min\{f(x)|x \in [u]_{\alpha}\} + \max\{f(x)|x \in [u]_{\alpha}\}}{2} - f(0) \\ \overline{f(u)} &= \frac{\max\{f(x)|x \in [u]_{\alpha}\} - \min\{f(x)|x \in [u]_{\alpha}\}}{2} - f(0). \end{aligned}$$

Similarly, starting with n valid pairs (\tilde{u}^i, \bar{u}^i) , $i = 1, 2, \dots, n$, and setting $\tilde{u} = (\tilde{u}^1, \dots, \tilde{u}^n) \in \mathbb{P}^n$, $\bar{u} = (\bar{u}^1, \dots, \bar{u}^n) \in (\mathbb{S}_0)^n$ we can construct two functions $\tilde{F} : \mathbb{P}^n \times (\mathbb{S}_0)^n \rightarrow \mathbb{P}$ and $\bar{F} : \mathbb{P}^n \times (\mathbb{S}_0)^n \rightarrow \mathbb{S}_0$ such that $(\tilde{F}(\tilde{u}, \bar{u}), \bar{F}(\tilde{u}, \bar{u}))$ is a valid pair.

A useful result is given by the following

Proposition 6. *Let $(\tilde{u}, \bar{u}), (\tilde{v}, \bar{v}) \in \mathbb{P} \times \mathbb{S}_0$ be two valid pairs.*

Then, $\forall a, b, c, d, e \in \mathbb{R}$ also

$$(a\tilde{u} + b\tilde{v} + c\tilde{u}\tilde{v} + d\bar{u}\bar{v} + e\tilde{u}\bar{v}, |a|\bar{u} + |b|\bar{v} + (|c| + |d| + |e|)\bar{u}\bar{v})$$

is a valid pair.

Proof. Consider $\alpha' < \alpha'' \in [0, 1]$; we have the following inequalities (remember that $\bar{u}_{\alpha'} \geq \bar{u}_{\alpha''} \geq 0$ and $\bar{v}_{\alpha'} \geq \bar{v}_{\alpha''} \geq 0$):

$$\begin{aligned} & |a(\tilde{u}_{\alpha''} - \tilde{u}_{\alpha'}) + b(\tilde{v}_{\alpha''} - \tilde{v}_{\alpha'}) + \\ & + c(\tilde{u}_{\alpha''}\tilde{v}_{\alpha''} - \tilde{u}_{\alpha'}\tilde{v}_{\alpha'}) + \\ & + d(\tilde{u}_{\alpha''}\bar{v}_{\alpha''} - \tilde{u}_{\alpha'}\bar{v}_{\alpha'}) + \\ & + e(\bar{u}_{\alpha''}\tilde{v}_{\alpha''} - \bar{u}_{\alpha'}\tilde{v}_{\alpha'})| \leq \\ & \leq |a||\tilde{u}_{\alpha''} - \tilde{u}_{\alpha'}| + |b||\tilde{v}_{\alpha''} - \tilde{v}_{\alpha'}| + \\ & + |c| [|\tilde{u}_{\alpha''}||\tilde{v}_{\alpha''} - \tilde{v}_{\alpha'}| + |\tilde{v}_{\alpha'}||\bar{u}_{\alpha''} - \bar{u}_{\alpha'}|] + \\ & + |d| [|\tilde{u}_{\alpha''}||\bar{v}_{\alpha''} - \bar{v}_{\alpha'}| + |\bar{v}_{\alpha'}||\bar{u}_{\alpha''} - \bar{u}_{\alpha'}|] + \\ & + |e| [\bar{u}_{\alpha''}|\tilde{v}_{\alpha''} - \tilde{v}_{\alpha'}| + |\tilde{v}_{\alpha'}||\bar{u}_{\alpha''} - \bar{u}_{\alpha'}|] \leq \\ & \leq |a|(\bar{u}_{\alpha'} - \bar{u}_{\alpha''}) + |b|(\bar{v}_{\alpha'} - \bar{v}_{\alpha''}) + \\ & + |c| \left[\bar{u}_{\alpha''}(\bar{v}_{\alpha'} - \bar{v}_{\alpha''}) + \bar{v}_{\alpha'}(\bar{u}_{\alpha'} - \bar{u}_{\alpha''}) \right] + \\ & + |d| \left[\bar{u}_{\alpha''}(\bar{v}_{\alpha'} - \bar{v}_{\alpha''}) + \bar{v}_{\alpha'}(\bar{u}_{\alpha'} - \bar{u}_{\alpha''}) \right] + \\ & + |e| \left[\bar{u}_{\alpha''}(\bar{v}_{\alpha'} - \bar{v}_{\alpha''}) + \bar{v}_{\alpha'}(\bar{u}_{\alpha'} - \bar{u}_{\alpha''}) \right] \leq \\ & \leq |a|(\bar{u}_{\alpha'} - \bar{u}_{\alpha''}) + |b|(\bar{v}_{\alpha'} - \bar{v}_{\alpha''}) + (|c| + |d| + |e|)(\bar{u}_{\alpha'}\bar{v}_{\alpha'} - \bar{u}_{\alpha''}\bar{v}_{\alpha''}). \end{aligned}$$

■

Also,

Proposition 7. (adapted from [[10]])

If (\tilde{u}, \bar{u}) and (\tilde{v}, \bar{v}) are two valid pairs, then also (\tilde{w}, \bar{w}) with

$$\begin{aligned} \tilde{w}_{\alpha} &= \frac{\min\{\tilde{u}_{\alpha} - \bar{u}_{\alpha}, \tilde{v}_{\alpha} - \bar{v}_{\alpha}\} + \max\{\tilde{u}_{\alpha} + \bar{u}_{\alpha}, \tilde{v}_{\alpha} + \bar{v}_{\alpha}\}}{2} \\ \bar{w}_{\alpha} &= \frac{\max\{\tilde{u}_{\alpha} + \bar{u}_{\alpha}, \tilde{v}_{\alpha} + \bar{v}_{\alpha}\} - \min\{\tilde{u}_{\alpha} - \bar{u}_{\alpha}, \tilde{v}_{\alpha} - \bar{v}_{\alpha}\}}{2} \end{aligned}$$

is a valid pair.

Finally, the Hausdorff distance on $\mathbb{F}_{\mathbb{I}}$ is defined by:

$$D(u, v) = \sup_{\alpha \in [0, 1]} \{ \max\{|u_{\alpha}^{-} - v_{\alpha}^{-}|, |u_{\alpha}^{+} - v_{\alpha}^{+}|\} \} \quad (13)$$

If we use the decomposition

$$\begin{cases} u_{\alpha}^{-} = \widehat{u}^{-} + \widetilde{u}_{\alpha} - \overline{u}_{\alpha} \\ u_{\alpha}^{+} = \widehat{u}^{+} + \widetilde{u}_{\alpha} + \overline{u}_{\alpha} \end{cases}$$

(and similarly for v), then

$$\begin{cases} u_{\alpha}^{-} - v_{\alpha}^{-} = (\widehat{u}^{-} - \widehat{v}^{-}) + (\widetilde{u}_{\alpha} - \widetilde{v}_{\alpha}) + (\overline{v}_{\alpha} - \overline{u}_{\alpha}) \\ u_{\alpha}^{+} - v_{\alpha}^{+} = (\widehat{u}^{+} - \widehat{v}^{+}) + (\widetilde{u}_{\alpha} - \widetilde{v}_{\alpha}) + (\overline{u}_{\alpha} - \overline{v}_{\alpha}) \end{cases};$$

we can obtain a modified distance on $\mathbb{F}_{\mathbb{I}}$ by considering the three components

$$\begin{cases} \widehat{D}(u, v) = \max \{ |\widehat{u}^{-} - \widehat{v}^{-}|, |\widehat{u}^{+} - \widehat{v}^{+}| \} \\ \widetilde{D}(u, v) = \sup_{\alpha \in [0,1]} |\widetilde{u}_{\alpha} - \widetilde{v}_{\alpha}| \\ \overline{D}(u, v) = \sup_{\alpha \in [0,1]} |\overline{u}_{\alpha} - \overline{v}_{\alpha}| \end{cases}$$

and by defining

$$D^{*}(u, v) = \widehat{D}(u, v) + \widetilde{D}(u, v) + \overline{D}(u, v). \quad (14)$$

It is possible to see that $D^{*}(u, v)$ is equivalent to $D(u, v)$ as $\widehat{D} \leq D$, $\widetilde{D} \leq 2D$, $\overline{D} \leq 2D$ and then $D \leq D^{*} \leq 5D$ (the proof in appendix).

3. GENERAL PROPERTIES OF ARITHMETIC OPERATIONS

In this section we analyze standard arithmetic operations between fuzzy numbers in terms of the decomposition

$$u = (\widehat{u}, \widetilde{u}, \overline{u}) \in \widehat{\mathbb{F}} \times \mathbb{P} \times \mathbb{S}_0 \quad (15)$$

and we study, in particular, distributivity and symmetry associated to multiplication and division.

Remember that (15) is equivalent to

$$\begin{cases} u_{\alpha}^{+} = \widehat{u} + \widetilde{u}_{\alpha} + \overline{u}_{\alpha} \\ u_{\alpha}^{-} = \widehat{u} + \widetilde{u}_{\alpha} - \overline{u}_{\alpha} \end{cases} \quad \forall \alpha \in [0, 1]. \quad (16)$$

3.1. Addition, scalar multiplication and subtraction. Addition, scalar multiplication and subtraction are immediate. Let

$$u = (\widehat{u}, \widetilde{u}, \overline{u}) \text{ and } v = (\widehat{v}, \widetilde{v}, \overline{v})$$

be two fuzzy numbers, then we can write:

$$u + v = (\widehat{u} + \widehat{v}, \widetilde{u} + \widetilde{v}, \overline{u} + \overline{v}); \quad (17)$$

$$ku = (k\widehat{u}, k\widetilde{u}, |k|\overline{u}), \text{ for } k \in \mathbb{R} \quad (18)$$

$$u - v = u + (-v) = (\widehat{u} - \widehat{v}, \widetilde{u} - \widetilde{v}, \overline{u} + \overline{v}). \quad (19)$$

It may be of interest to note that addition and difference have the same 0-symmetric fuzzy components $\overline{u} + \overline{v}$ and they differ (only) in the crisp and the profile parts. On the other hand, it is well known that:

$$u = (0, 0, \overline{u}) \in \mathbb{S}_0 \iff u = -u$$

and that

$$u, v \in \mathbb{S}_0 \iff u + v = u - v = v - u = -u - v.$$

These facts can be a little formalized; consider the following relation on \mathbb{F} ($u \sim v$ if they differ only for the \mathbb{S}_0 -component):

$$u \sim v \iff \begin{cases} \hat{u} = \hat{v} \\ \tilde{u}_\alpha = \tilde{v}_\alpha \quad \forall \alpha \in [0, 1] \end{cases}$$

which is an equivalence relation on \mathbb{F} . The equivalence class corresponding to $0 = (0, 0, 0) \in \mathbb{S}_0$ (the null element of addition in \mathbb{F}) is the whole \mathbb{S}_0 , i.e. $[0]_\sim = \mathbb{S}_0$; in fact, with respect to \sim all the elements of \mathbb{S}_0 are equivalent to 0 and in particular $(u - u) \sim 0$ $\forall u \in \mathbb{F}$, as $(u - u) = (\hat{u}, \tilde{u}, \bar{u}) - (\hat{u}, \tilde{u}, \bar{u}) = (0, 0, 2\bar{u}) \in \mathbb{S}_0$.

3.2. Multiplication. Consider now the multiplication of $u, v \in \mathbb{F}$ where $u = (\hat{u}, \tilde{u}, \bar{u})$ and $v = (\hat{v}, \tilde{v}, \bar{v})$ are fuzzy numbers with

$$\begin{cases} u_\alpha^+ = \hat{u} + \tilde{u}_\alpha + \bar{u}_\alpha \\ u_\alpha^- = \hat{u} + \tilde{u}_\alpha - \bar{u}_\alpha \end{cases} \quad \text{and} \quad \begin{cases} v_\alpha^+ = \hat{v} + \tilde{v}_\alpha + \bar{v}_\alpha \\ v_\alpha^- = \hat{v} + \tilde{v}_\alpha - \bar{v}_\alpha \end{cases}$$

The α -cuts of the fuzzy product uv are given (see the Appendix) by:

$$\begin{aligned} [uv]_\alpha &= [u_\alpha^-, u_\alpha^+] [v_\alpha^-, v_\alpha^+] \\ &= [\min \{ (\hat{u} + \tilde{u}_\alpha) v_\alpha^- - \bar{u}_\alpha |v_\alpha^-|, (\hat{u} + \tilde{u}_\alpha) v_\alpha^+ - \bar{u}_\alpha |v_\alpha^+| \}, \\ &\quad \max \{ (\hat{u} + \tilde{u}_\alpha) v_\alpha^- + \bar{u}_\alpha |v_\alpha^-|, (\hat{u} + \tilde{u}_\alpha) v_\alpha^+ + \bar{u}_\alpha |v_\alpha^+| \}]. \end{aligned} \quad (20)$$

Distributivity of multiplication. We use (20) to analyze the distributivity property of the multiplication.

Given three fuzzy numbers in terms of their components

$$u = (\hat{u}, \tilde{u}, \bar{u}); \quad v = (\hat{v}, \tilde{v}, \bar{v}); \quad z = (\hat{z}, \tilde{z}, \bar{z})$$

we obtain from (20) the α -cuts

$$[(u + v)z]_\alpha = \left[\min \left\{ \begin{aligned} &(\hat{u} + \hat{v} + \tilde{u}_\alpha + \tilde{v}_\alpha) z_\alpha^- - (\bar{u}_\alpha + \bar{v}_\alpha) |z_\alpha^-|, \\ &(\hat{u} + \hat{v} + \tilde{u}_\alpha + \tilde{v}_\alpha) z_\alpha^+ - (\bar{u}_\alpha + \bar{v}_\alpha) |z_\alpha^+| \end{aligned} \right\}, \right. \\ \left. \max \left\{ \begin{aligned} &(\hat{u} + \hat{v} + \tilde{u}_\alpha + \tilde{v}_\alpha) z_\alpha^- - (\bar{u}_\alpha + \bar{v}_\alpha) |z_\alpha^-|, \\ &(\hat{u} + \hat{v} + \tilde{u}_\alpha + \tilde{v}_\alpha) z_\alpha^+ + (\bar{u}_\alpha + \bar{v}_\alpha) |z_\alpha^+| \end{aligned} \right\} \right]$$

and

$$[uz + vz]_\alpha = \left[\begin{aligned} &\min \{ (\hat{u} + \tilde{u}_\alpha) z_\alpha^- - \bar{u}_\alpha |z_\alpha^-|, (\hat{u} + \tilde{u}_\alpha) z_\alpha^+ - \bar{u}_\alpha |z_\alpha^+| \} \\ &+ \min \{ (\hat{v} + \tilde{v}_\alpha) z_\alpha^- - \bar{v}_\alpha |z_\alpha^-|, (\hat{v} + \tilde{v}_\alpha) z_\alpha^+ - \bar{v}_\alpha |z_\alpha^+| \}, \\ &\max \{ (\hat{u} + \tilde{u}_\alpha) z_\alpha^- + \bar{u}_\alpha |z_\alpha^-|, (\hat{u} + \tilde{u}_\alpha) z_\alpha^+ + \bar{u}_\alpha |z_\alpha^+| \} \\ &+ \max \{ (\hat{v} + \tilde{v}_\alpha) z_\alpha^- + \bar{v}_\alpha |z_\alpha^-|, (\hat{v} + \tilde{v}_\alpha) z_\alpha^+ + \bar{v}_\alpha |z_\alpha^+| \} \end{aligned} \right].$$

We then have:

Theorem 8. *The distributivity $(u + v)z = uz + vz$ is valid if and only if, for each $\alpha \in [0, 1]$, at least one of the following four systems of inequalities is satisfied:*

$$\begin{cases} (\hat{u} + \tilde{u}_\alpha) z_\alpha^- - \bar{u}_\alpha |z_\alpha^-| \leq (\hat{u} + \tilde{u}_\alpha) z_\alpha^+ - \bar{u}_\alpha |z_\alpha^+| \\ (\hat{v} + \tilde{v}_\alpha) z_\alpha^- - \bar{v}_\alpha |z_\alpha^-| \leq (\hat{v} + \tilde{v}_\alpha) z_\alpha^+ - \bar{v}_\alpha |z_\alpha^+| \\ (\hat{u} + \tilde{u}_\alpha) z_\alpha^+ + \bar{u}_\alpha |z_\alpha^+| \leq (\hat{u} + \tilde{u}_\alpha) z_\alpha^- + \bar{u}_\alpha |z_\alpha^-| \\ (\hat{v} + \tilde{v}_\alpha) z_\alpha^+ + \bar{v}_\alpha |z_\alpha^+| \leq (\hat{v} + \tilde{v}_\alpha) z_\alpha^- + \bar{v}_\alpha |z_\alpha^-| \end{cases} \quad (21)$$

$$\left\{ \begin{array}{l} (\hat{u} + \tilde{u}_\alpha)z_\alpha^- - \bar{u}_\alpha |z_\alpha^-| \leq (\hat{u} + \tilde{u}_\alpha)z_\alpha^+ - \bar{u}_\alpha |z_\alpha^+| \\ (\hat{v} + \tilde{v}_\alpha)z_\alpha^- - \bar{v}_\alpha |z_\alpha^-| \leq (\hat{v} + \tilde{v}_\alpha)z_\alpha^+ - \bar{v}_\alpha |z_\alpha^+| \\ (\hat{u} + \tilde{u}_\alpha)z_\alpha^- + \bar{u}_\alpha |z_\alpha^-| \leq (\hat{u} + \tilde{u}_\alpha)z_\alpha^+ + \bar{u}_\alpha |z_\alpha^+| \\ (\hat{v} + \tilde{v}_\alpha)z_\alpha^- + \bar{v}_\alpha |z_\alpha^-| \leq (\hat{v} + \tilde{v}_\alpha)z_\alpha^+ + \bar{v}_\alpha |z_\alpha^+| \end{array} \right. \quad (22)$$

$$\left\{ \begin{array}{l} (\hat{u} + \tilde{u}_\alpha)z_\alpha^+ - \bar{u}_\alpha |z_\alpha^+| \leq (\hat{u} + \tilde{u}_\alpha)z_\alpha^- - \bar{u}_\alpha |z_\alpha^-| \\ (\hat{v} + \tilde{v}_\alpha)z_\alpha^+ - \bar{v}_\alpha |z_\alpha^+| \leq (\hat{v} + \tilde{v}_\alpha)z_\alpha^- - \bar{v}_\alpha |z_\alpha^-| \\ (\hat{u} + \tilde{u}_\alpha)z_\alpha^+ + \bar{u}_\alpha |z_\alpha^+| \leq (\hat{u} + \tilde{u}_\alpha)z_\alpha^- + \bar{u}_\alpha |z_\alpha^-| \\ (\hat{v} + \tilde{v}_\alpha)z_\alpha^+ + \bar{v}_\alpha |z_\alpha^+| \leq (\hat{v} + \tilde{v}_\alpha)z_\alpha^- + \bar{v}_\alpha |z_\alpha^-| \end{array} \right. \quad (23)$$

$$\left\{ \begin{array}{l} (\hat{u} + \tilde{u}_\alpha)z_\alpha^+ - \bar{u}_\alpha |z_\alpha^+| \leq (\hat{u} + \tilde{u}_\alpha)z_\alpha^- - \bar{u}_\alpha |z_\alpha^-| \\ (\hat{v} + \tilde{v}_\alpha)z_\alpha^+ - \bar{v}_\alpha |z_\alpha^+| \leq (\hat{v} + \tilde{v}_\alpha)z_\alpha^- - \bar{v}_\alpha |z_\alpha^-| \\ (\hat{u} + \tilde{u}_\alpha)z_\alpha^- + \bar{u}_\alpha |z_\alpha^-| \leq (\hat{u} + \tilde{u}_\alpha)z_\alpha^+ + \bar{u}_\alpha |z_\alpha^+| \\ (\hat{v} + \tilde{v}_\alpha)z_\alpha^- + \bar{v}_\alpha |z_\alpha^-| \leq (\hat{v} + \tilde{v}_\alpha)z_\alpha^+ + \bar{v}_\alpha |z_\alpha^+| \end{array} \right. \quad (24)$$

Proof. (see Appendix) ■

We have the following dependencies:

- (21) $\implies |z_\alpha^-| \geq |z_\alpha^+|$;
- (22) $\implies \hat{u} + \tilde{u}_\alpha \geq 0, \hat{v} + \tilde{v}_\alpha \geq 0$;
- (23) $\implies \hat{u} + \tilde{u}_\alpha \leq 0, \hat{v} + \tilde{v}_\alpha \leq 0$;
- (24) $\implies |z_\alpha^-| \leq |z_\alpha^+|$.

The following propositions give relevant cases of validity of the distributivity and are immediately verified (at our knowledge, some of them are new in the fuzzy literature).

The first two properties say that the distributivity is valid if $z \in \mathbb{S}_0$ and, independently of their sign, the fuzzy numbers u and v have "homogeneous" asymmetries, i.e. $(\hat{u} + \tilde{u}_\alpha)(\hat{v} + \tilde{v}_\alpha) \geq 0$.

Proposition 9. Given $z \in \mathbb{S}_0$ and $\hat{u} + \tilde{u}_\alpha \geq 0, \hat{v} + \tilde{v}_\alpha \geq 0$ then system (22) is satisfied.

Proof. In fact, $|z_\alpha^-| = -z_\alpha^- = z_\alpha^+ > 0$ and (22) reduces simply to $(\hat{u} + \tilde{u}_\alpha)z_\alpha^+ \geq 0$ and $(\hat{v} + \tilde{v}_\alpha)z_\alpha^+ \geq 0$. ■

Proposition 10. Given $z \in \mathbb{S}_0$ and $\hat{u} + \tilde{u}_\alpha \leq 0, \hat{v} + \tilde{v}_\alpha \leq 0$ then system (23) is satisfied.

Proof. In fact, $|z_\alpha^-| = -z_\alpha^- = z_\alpha^+ > 0$ and (23) reduces simply to $(\hat{u} + \tilde{u}_\alpha)z_\alpha^+ \leq 0$ and $(\hat{v} + \tilde{v}_\alpha)z_\alpha^+ \leq 0$. ■

Proposition 11. Given $z \in \mathbb{F}_+$ and $u, v \in \mathbb{F}_0$ then system (24) is satisfied.

Proof. In this case, z has $z_\alpha^- = |z_\alpha^-|, z_\alpha^+ = |z_\alpha^+|$, and $z_\alpha^- \leq z_\alpha^+$; u and v have $-\bar{u}_\alpha \leq \hat{u} + \tilde{u}_\alpha \leq \bar{u}_\alpha$ and $-\bar{v}_\alpha \leq \hat{v} + \tilde{v}_\alpha \leq \bar{v}_\alpha$; then system (24) is immediate. ■

Proposition 12. Given $z \in \mathbb{F}_-$ and $u, v \in \mathbb{F}_0$ then system (21) is satisfied.

Proof. In this case, z has $z_\alpha^- = -|z_\alpha^-|, z_\alpha^+ = -|z_\alpha^+|$, and $z_\alpha^- \leq z_\alpha^+$; u and v have $-\bar{u}_\alpha \leq \hat{u} + \tilde{u}_\alpha \leq \bar{u}_\alpha$ and $-\bar{v}_\alpha \leq \hat{v} + \tilde{v}_\alpha \leq \bar{v}_\alpha$; then system (21) is immediate. ■

Proposition 13. ([11]) Given $z \in \mathbb{F}$ and $u, v \in \mathbb{S}_0$ then either system (21) or (24) is satisfied.

Proposition 14. ([11]) Given $z \in \mathbb{F}$ and $u, v \in \mathbb{F}_+$ then system (22) is satisfied.

Proposition 15. ([11]) Given $z \in \mathbb{F}$ and $u, v \in \mathbb{F}_-$ then system (23) is satisfied.

So, the cases where the distributivity of multiplication is valid require that at least one of the two factors are of special nature; in particular, distributivity is valid within $\mathbb{S}_0, \mathbb{F}_+$ or \mathbb{F}_- .

Symmetry in Multiplication. By the use of the decomposition (11) we can analyze the effect of the multiplication on the symmetry of the product fuzzy number.

It is immediate to see that if $u, v \in \mathbb{S}_0$ i.e. $u = (0, 0, \bar{u})$ and $v = (0, 0, \bar{v})$, then $uv = (0, 0, \bar{u}\bar{v}) \in \mathbb{S}_0$ is the exact product.

If we take $u, v \in \mathbb{F}_0$ i.e. $u = (0, \tilde{u}, \bar{u})$ and $v = (0, \tilde{v}, \bar{v})$ then the α -cuts of the product $uv \in \mathbb{F}_0$ are given by

$$\begin{aligned} [uv]_\alpha &= [\tilde{u}_\alpha \tilde{v}_\alpha - \bar{u}_\alpha \bar{v}_\alpha - |\bar{u}_\alpha \tilde{v}_\alpha - \tilde{u}_\alpha \bar{v}_\alpha|, \\ &\quad \tilde{u}_\alpha \tilde{v}_\alpha + \bar{u}_\alpha \bar{v}_\alpha + |\bar{u}_\alpha \tilde{v}_\alpha + \tilde{u}_\alpha \bar{v}_\alpha|]; \end{aligned} \quad (25)$$

it follows that the elements of the decomposition of uv are:

$$\begin{aligned} (\widetilde{uv})_\alpha &= \tilde{u}_\alpha \tilde{v}_\alpha + \frac{|\bar{u}_\alpha \tilde{v}_\alpha + \tilde{u}_\alpha \bar{v}_\alpha| - |\bar{u}_\alpha \tilde{v}_\alpha - \tilde{u}_\alpha \bar{v}_\alpha|}{2} \\ &= \tilde{u}_\alpha \tilde{v}_\alpha + \max\{\bar{u}_\alpha \tilde{v}_\alpha, \tilde{u}_\alpha \bar{v}_\alpha\} \end{aligned}$$

and

$$\begin{aligned} \overline{(uv)}_\alpha &= \bar{u}_\alpha \bar{v}_\alpha + \frac{|\bar{u}_\alpha \tilde{v}_\alpha - \tilde{u}_\alpha \bar{v}_\alpha| + |\bar{u}_\alpha \tilde{v}_\alpha + \tilde{u}_\alpha \bar{v}_\alpha|}{2} \\ &= \bar{u}_\alpha \bar{v}_\alpha + w_\alpha. \end{aligned}$$

where $w_\alpha \in \{\bar{u}_\alpha \tilde{v}_\alpha, \tilde{u}_\alpha \bar{v}_\alpha, -\bar{u}_\alpha \tilde{v}_\alpha, -\tilde{u}_\alpha \bar{v}_\alpha\}$

and we see how the profile functions of u and v contribute to the definition of the product.

We also see from (25) that \mathbb{S}_0 absorbs \mathbb{F}_0 in the sense that if $u \in \mathbb{S}_0$ and $v \in \mathbb{F}_0$ then $uv = (0, 0, \bar{u}\bar{v} + |\bar{u}\tilde{v}|) \in \mathbb{S}_0$ (in a similar way, \mathbb{F}_0 absorbs \mathbb{F} as if $u \in \mathbb{F}_0$, $v \in \mathbb{F}$ then $uv \in \mathbb{F}_0$).

Consider now the case $u, v \in \mathbb{S}$ i.e. $u = (\hat{u}, 0, \bar{u})$ and $v = (\hat{v}, 0, \bar{v})$. Then the product uv is given (after some algebra) by

$$\begin{aligned} [uv]_\alpha &= \hat{u}\hat{v} + [\min\{\bar{u}_\alpha \bar{v}_\alpha - |\hat{u}\bar{v}_\alpha + \bar{u}_\alpha \hat{v}|, -\bar{u}_\alpha \bar{v}_\alpha - |\hat{u}\bar{v}_\alpha - \bar{u}_\alpha \hat{v}|\}, \\ &\quad \max\{\bar{u}_\alpha \bar{v}_\alpha + |\hat{u}\bar{v}_\alpha + \bar{u}_\alpha \hat{v}|, -\bar{u}_\alpha \bar{v}_\alpha + |\hat{u}\bar{v}_\alpha - \bar{u}_\alpha \hat{v}|\}]. \end{aligned} \quad (26)$$

Note that, in this case, $\min\{.,.\} \leq 0$ and $\max\{.,.\} \geq 0$ and if $\min\{.,.\} = \bar{u}_\alpha \bar{v}_\alpha - |\hat{u}\bar{v}_\alpha + \bar{u}_\alpha \hat{v}|$ then $\max\{.,.\} = \bar{u}_\alpha \bar{v}_\alpha + |\hat{u}\bar{v}_\alpha + \bar{u}_\alpha \hat{v}|$.

If we compute $(\widetilde{uv})_\alpha = \frac{\max_\alpha + \min_\alpha}{2}$ and $(\overline{uv})_\alpha = \frac{\max_\alpha - \min_\alpha}{2}$ we obtain the following three cases:

Case 1. $\min\{.,.\} = \bar{u}_\alpha \bar{v}_\alpha - |\hat{u}\bar{v}_\alpha + \bar{u}_\alpha \hat{v}|$ and $\max\{.,.\} = \bar{u}_\alpha \bar{v}_\alpha + |\hat{u}\bar{v}_\alpha + \bar{u}_\alpha \hat{v}|$: then $(\widetilde{uv})_\alpha = \bar{u}_\alpha \bar{v}_\alpha$ and $(\overline{uv})_\alpha = |\hat{u}\bar{v}_\alpha + \bar{u}_\alpha \hat{v}|$ and a positive asymmetry is introduced by the operation;

Case 2. $\min\{.,.\} = -\bar{u}_\alpha \bar{v}_\alpha - |\hat{u}\bar{v}_\alpha - \bar{u}_\alpha \hat{v}|$ and $\max\{.,.\} = -\bar{u}_\alpha \bar{v}_\alpha + |\hat{u}\bar{v}_\alpha - \bar{u}_\alpha \hat{v}|$: then $(\widetilde{uv})_\alpha = -\bar{u}_\alpha \bar{v}_\alpha$ and $(\overline{uv})_\alpha = |\hat{u}\bar{v}_\alpha + \bar{u}_\alpha \hat{v}|$ and a negative asymmetry is introduced;

Case 3. $\min\{.,.\} = -\bar{u}_\alpha \bar{v}_\alpha - |\hat{u}\bar{v}_\alpha - \bar{u}_\alpha \hat{v}|$ and $\max\{.,.\} = \bar{u}_\alpha \bar{v}_\alpha + |\hat{u}\bar{v}_\alpha + \bar{u}_\alpha \hat{v}|$: then $(\widetilde{uv})_\alpha = |\hat{u}\bar{v}_\alpha + \bar{u}_\alpha \hat{v}| - |\hat{u}\bar{v}_\alpha - \bar{u}_\alpha \hat{v}|$ and $(\overline{uv})_\alpha = \bar{u}_\alpha \bar{v}_\alpha + \frac{1}{2}(|\hat{u}\bar{v}_\alpha + \bar{u}_\alpha \hat{v}| + |\hat{u}\bar{v}_\alpha - \bar{u}_\alpha \hat{v}|)$ and an unsigned asymmetry is introduced.

3.3. Division. We complete this section by analyzing the division between fuzzy numbers $u \in \mathbb{F}$ and $z \in \mathbb{F}_- \cup \mathbb{F}_+$.

The reciprocal of z is the fuzzy number given by z^{-1} with α -cuts

$$[z^{-1}]_\alpha = \left[\frac{1}{z_\alpha^+}, \frac{1}{z_\alpha^-} \right]$$

and the division is defined as

$$\frac{u}{z} = uz^{-1}.$$

Note that if $u = (0, 0, \bar{u}_\alpha) \in \mathbb{S}_0$ then $\frac{u}{z} \in \mathbb{S}_0$ as the α -cuts are

$$\left[\frac{u}{z} \right]_\alpha = \left[-\bar{u}_\alpha \max \left\{ \frac{1}{z_\alpha^+}, -\frac{1}{z_\alpha^-} \right\}, \bar{u}_\alpha \max \left\{ \frac{1}{z_\alpha^+}, -\frac{1}{z_\alpha^-} \right\} \right].$$

If $z = (\hat{z}, 0, \bar{z}) \in \mathbb{S}_- \cup \mathbb{S}_+$ is symmetric, then it is easy to see that $z^{-1} = (\hat{z}^{-1}, \widetilde{z^{-1}}, \overline{z^{-1}})$ is not symmetric as

$$\begin{aligned} \widetilde{z_\alpha^{-1}} &= \frac{\bar{z}_\alpha^2}{\hat{z}(\hat{z}^2 - \bar{z}_\alpha^2)} \\ \overline{z_\alpha^{-1}} &= \frac{\bar{z}_\alpha}{\hat{z}^2 - \bar{z}_\alpha^2}. \end{aligned}$$

The distributivity of the division, i.e. the equality

$$\frac{u+v}{z} = \frac{u}{z} + \frac{v}{z}$$

can be written in terms of the distributivity of the multiplication:

$$(u+v)z^{-1} = uz^{-1} + vz^{-1}.$$

Proposition 16. *If $z \in \mathbb{F}_+ \cup \mathbb{F}_-$ and $u, v \in \mathbb{F}_0$ (or in particular $u, v \in \mathbb{S}_0$) then the division is distributive.*

Proof. *In fact, if $z \in \mathbb{F}_+$ then $z^{-1} \in \mathbb{F}_+$ or if $z \in \mathbb{F}_-$ then $z^{-1} \in \mathbb{F}_-$; the application of propositions 13 and 14 gives the distributivity. ■*

Proposition 17. ([11]) *If $z \in \mathbb{F}_+ \cup \mathbb{F}_-$ and $u, v \in \mathbb{F}_+$ (or $u, v \in \mathbb{F}_-$) then the division is distributive.*

4. APPROXIMATE DISTRIBUTIVE OPERATIONS

In this section we will use the decomposition of the fuzzy numbers introduced in section 2 and the concept of valid pair, to define approximate multiplication and division; in particular, we will focus on approximate operations with the distributivity property.

4.1. Approximate Multiplication. By the help of the propositions in section 2. it is immediate to verify that, given two fuzzy numbers $u = (\hat{u}, \tilde{u}, \bar{u})$ and $v = (\hat{v}, \tilde{v}, \bar{v})$, then also $(\hat{u}\hat{v}, \tilde{u}\tilde{v}, \bar{u}\bar{v})$ is a valid fuzzy number (use $a = b = d = e = 0$ and $c = 1$) ..

Definition 18. *Define an approximate multiplication as*

$$[u * v]_\alpha = [\hat{u}\hat{v} + \tilde{u}_\alpha\tilde{v}_\alpha - \bar{u}_\alpha\bar{v}_\alpha, \hat{u}\hat{v} + \tilde{u}_\alpha\tilde{v}_\alpha + \bar{u}_\alpha\bar{v}_\alpha]$$

or, in terms of the decomposition,

$$u * v = (\hat{u}\hat{v}, \tilde{u}\tilde{v}, \bar{u}\bar{v}).$$

We will now compare the standard multiplication with the new multiplication above; considering the standard product uv , after some algebra, we can write its α -cuts as

$$[uv]_\alpha = \widehat{u}\widehat{v} + \widehat{u}\widetilde{v}_\alpha + \widetilde{v}_\alpha\widehat{u} + \widetilde{u}_\alpha\widetilde{v}_\alpha + A_\alpha$$

where

$$A_\alpha = \left[\min \left\{ \begin{array}{l} \overline{u}_\alpha\overline{v}_\alpha - |(\widehat{u} + \widetilde{u}_\alpha)\overline{v}_\alpha + (\widehat{v} + \widetilde{v}_\alpha)\overline{u}_\alpha|, \\ -\overline{u}_\alpha\overline{v}_\alpha - |(\widehat{u} + \widetilde{u}_\alpha)\overline{v}_\alpha - (\widehat{v} + \widetilde{v}_\alpha)\overline{u}_\alpha| \end{array} \right\}, \right. \\ \left. \max \left\{ \begin{array}{l} \overline{u}_\alpha\overline{v}_\alpha + |(\widehat{u} + \widetilde{u}_\alpha)\overline{v}_\alpha + (\widehat{v} + \widetilde{v}_\alpha)\overline{u}_\alpha|, \\ -\overline{u}_\alpha\overline{v}_\alpha + |(\widehat{u} + \widetilde{u}_\alpha)\overline{v}_\alpha - (\widehat{v} + \widetilde{v}_\alpha)\overline{u}_\alpha| \end{array} \right\} \right].$$

It also holds:

$$\begin{aligned} [u * v]_\alpha &= \widehat{u}\widehat{v} + \widetilde{u}_\alpha\widetilde{v}_\alpha + B_\alpha \\ \text{where } B_\alpha &= [-\overline{u}_\alpha\overline{v}_\alpha, \overline{u}_\alpha\overline{v}_\alpha]. \end{aligned}$$

Note that, in any case, $B_\alpha \subseteq A_\alpha$.

Case 19. (\mathbb{S}_0 - case)

If $u, v \in \mathbb{S}_0$ then $\forall \alpha \in [0, 1]$ the following is immediate:

$$(u * v)_\alpha = (uv)_\alpha.$$

Case 20. (\mathbb{S} - case)

If $u, v \in \mathbb{S}$ then $\forall \alpha \in [0, 1]$ the following is true:

$$(u * v)_\alpha \subseteq (uv)_\alpha.$$

Proof. In this case we have that $\widetilde{u}_\alpha = \widetilde{v}_\alpha = 0 \quad \forall \alpha$ then $(uv)_\alpha = \widehat{u}\widehat{v} + A_\alpha$ and $(u * v)_\alpha = \widehat{u}\widehat{v} + B_\alpha$. Then $(u * v)_\alpha \subseteq (uv)_\alpha$ as $B_\alpha \subseteq A_\alpha$. ■

Case 21. (\mathbb{F}_0 - case)

If $u, v \in \mathbb{F}_0$ then $\forall \alpha \in [0, 1]$ we have

$$(u * v)_\alpha \subseteq (uv)_\alpha$$

Proof. As $\widehat{u} = \widehat{v} = 0$ it follows that $u_\alpha^- = \widetilde{u}_\alpha - \overline{u}_\alpha \leq 0$, $v_\alpha^- = \widetilde{v}_\alpha - \overline{v}_\alpha \leq 0$, $u_\alpha^+ = \widetilde{u}_\alpha + \overline{u}_\alpha \geq 0$ and $v_\alpha^+ = \widetilde{v}_\alpha + \overline{v}_\alpha \geq 0$. In this case

$$(uv)_\alpha = [\min \{u_\alpha^- v_\alpha^+, u_\alpha^+ v_\alpha^-\}, \max \{u_\alpha^- v_\alpha^-, u_\alpha^+ v_\alpha^+\}]$$

Consider the following averages of the terms in the min and the max above

$$w_\alpha^- = \frac{u_\alpha^- v_\alpha^+ + u_\alpha^+ v_\alpha^-}{2}, \quad w_\alpha^+ = \frac{u_\alpha^- v_\alpha^- + u_\alpha^+ v_\alpha^+}{2};$$

it is obvious that $[w_\alpha^-, w_\alpha^+] \subseteq (uv)_\alpha$ as $\min \leq \text{average} \leq \max$. We now see that $(u * v)_\alpha = [w_\alpha^-, w_\alpha^+]$; in fact, by computing the decomposition of the fuzzy number w defined by the α -cuts $[w_\alpha^-, w_\alpha^+]$ we find $\widetilde{w}_\alpha = \frac{w_\alpha^- + w_\alpha^+}{2} = \frac{u_\alpha^- + u_\alpha^+}{2} \cdot \frac{v_\alpha^- + v_\alpha^+}{2} = \widetilde{u}_\alpha \widetilde{v}_\alpha$ and $\overline{w}_\alpha = \frac{w_\alpha^+ - w_\alpha^-}{2} = \frac{u_\alpha^+ - u_\alpha^-}{2} \cdot \frac{v_\alpha^+ - v_\alpha^-}{2} = \overline{u}_\alpha \overline{v}_\alpha$. ■

Case 22. If $u \in \mathbb{S}_0$, $v \in \mathbb{F}$ then

$$(u * v)_\alpha \subseteq (uv)_\alpha.$$

Proof. In fact, as $\widehat{u} = 0$ and $\widetilde{u}_\alpha = 0 \forall \alpha$, then

$$\begin{aligned} (uv)_\alpha &= [\min \{\overline{u}_\alpha \overline{v}_\alpha - |\widehat{v} + \widetilde{v}_\alpha| \overline{u}_\alpha, -\overline{u}_\alpha \overline{v}_\alpha - |\widehat{v} + \widetilde{v}_\alpha| \overline{u}_\alpha\}, \\ &\quad \max \{\overline{u}_\alpha \overline{v}_\alpha + |\widehat{v} + \widetilde{v}_\alpha| \overline{u}_\alpha, -\overline{u}_\alpha \overline{v}_\alpha + |\widehat{v} + \widetilde{v}_\alpha| \overline{u}_\alpha\}] \\ &= [-\overline{u}_\alpha \overline{v}_\alpha - |\widehat{v} + \widetilde{v}_\alpha| \overline{u}_\alpha, \overline{u}_\alpha \overline{v}_\alpha + |\widehat{v} + \widetilde{v}_\alpha| \overline{u}_\alpha] \in \mathbb{S}_0 \end{aligned}$$

and $(u * v)_\alpha = [-\overline{u}_\alpha \overline{v}_\alpha, \overline{u}_\alpha \overline{v}_\alpha]$ so that $(u * v)_\alpha \subseteq (uv)_\alpha$. ■

The properties above suggest that the fuzzy number

$$u * v = (\widehat{u}\widehat{v}, \widetilde{u}\widetilde{v}, \overline{u}\overline{v})$$

can be used as an approximation of the standard multiplication uv .

In particular, the approximation is exact if $u, v \in \mathbb{S}_0$ and it reduces the range of the fuzziness (i.e. the range of the α -cuts) when $u, v \in \mathbb{S}$, when $u, v \in \mathbb{F}_0$ and when $u \in \mathbb{S}_0, v \in \mathbb{F}$.

Proposition 23. (Distributivity of $*$)

If $u, v, z \in \mathbb{F}$ then $(u + v) * z = u * z + v * z$.

Proof. In fact $(u + v) * z = ((\widehat{u} + \widehat{v}) * \widehat{z}, (\widetilde{u} + \widetilde{v}) * \widetilde{z}, (\overline{u} + \overline{v}) * \overline{z}) = u * z + v * z$ as distributivity is valid for each of the three components. ■

4.2. Approximate division. In a similar way as we have seen for the multiplication, we can apply the decomposition $u = (\widehat{u}, \widetilde{u}, \overline{u})$ to the division u/v where $v \in \mathbb{F}_-$ or $v \in \mathbb{F}_+$. Note that, for such v we always have $\widehat{v} \neq 0$.

In the special case where $u = (1, 0, 0)$ we obtain the fuzzy reciprocal of v , having the following α -cuts :

$$(v^{-1})_\alpha = \left[\frac{1}{v_\alpha^+}, \frac{1}{v_\alpha^-} \right], \forall \alpha \in [0, 1].$$

The decomposition of v^{-1} is then given by

$$\widehat{v^{-1}} = \frac{1}{\widehat{v}}, \widetilde{v_\alpha^{-1}} = \frac{\widehat{v} + \widetilde{v}_\alpha}{v_\alpha^- v_\alpha^+} - \frac{1}{\widehat{v}} \text{ and } \overline{v_\alpha^{-1}} = \frac{\overline{v}_\alpha}{v_\alpha^- v_\alpha^+}.$$

Using the results that we have seen for the multiplication, $(\widetilde{u}v^{-1}, \overline{u}\overline{v^{-1}})$ is a valid pair and we can approximate the division by using it:

Definition 24. For $u = (\widehat{u}, \widetilde{u}, \overline{u}) \in \mathbb{F}$ and $v = (\widehat{v}, \widetilde{v}, \overline{v}) \in \mathbb{F}_- \cup \mathbb{F}_+$ define an approximate division as

$$u : v = u * v^{-1} = (\widehat{u}, \widetilde{u}v^{-1}, \overline{u}\overline{v^{-1}}).$$

The following properties are then immediate:

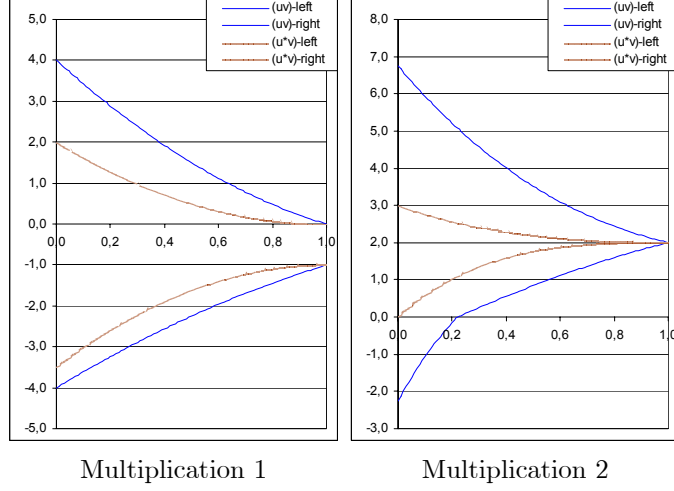
1. $u \in \mathbb{S}_0 \implies (u : v)_\alpha \subseteq (\frac{u}{v})_\alpha \forall \alpha$;
2. $(u + v) : z = u : z + v : z$.

Note also that, if $v \in \mathbb{S}_- \cup \mathbb{S}_+$, then $\widetilde{v_\alpha^{-1}} = \frac{\overline{v}_\alpha^2}{\widehat{v}^2 - \overline{v}_\alpha^2} \neq 0$ so that $v^{-1} \notin \mathbb{S}$ (the reciprocal operation introduces a form of asymmetry as a nonzero profile).

5. EXAMPLES

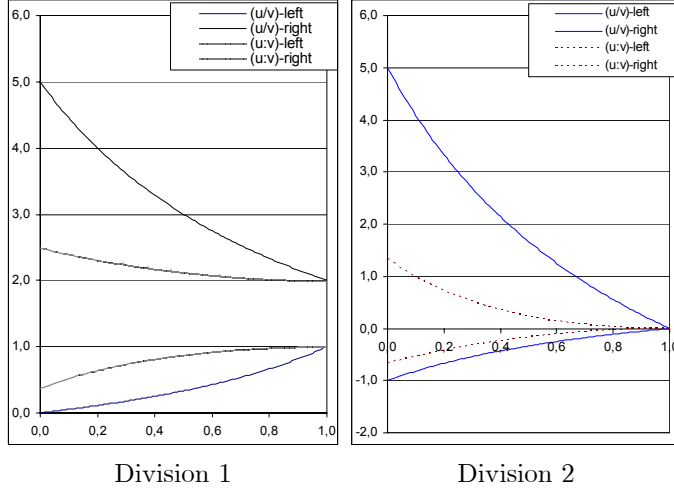
In Multiplication 1, the two fuzzy numbers u and v are both trapezoidal with linear left and right branches $u = \langle -2, 0, 1, 2 \rangle$ and $v = \langle -2, -1, 0, 2 \rangle$.

In Multiplication 2, we use u and v in the decomposed form with $\hat{u} = 1$, $\tilde{u}_\alpha = -(1 - \alpha)^2/2$, $\bar{u}_\alpha = (1 - \alpha)(2 - \alpha)/2$ and $\hat{v} = 2$, $\tilde{v}_\alpha = -2\tilde{u}_\alpha$, $\bar{v}_\alpha = 1.5\bar{u}_\alpha$.



In Division 1, the two fuzzy numbers u and v are both trapezoidal with linear left and right branches $u = \langle 1, 2, 3, 4 \rangle$ and $v = \langle 2, 3, 5, 7 \rangle$.

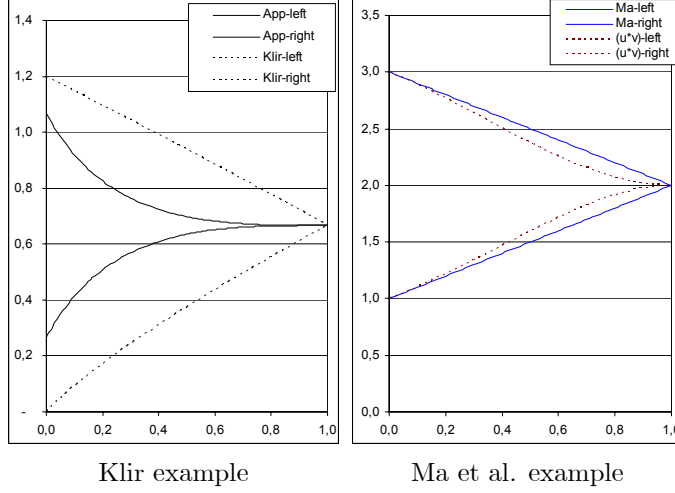
In Division 2, the two fuzzy numbers u and v are both trapezoidal with linear left and right branches $u = \langle -1, 0, 0, 5 \rangle$ and $v = \langle 1, 2, 2, 3 \rangle$. Note that $u \in \mathbb{F}_0$.



In the calculus of fuzzy expressions we take an example with triangular $u = \langle 0, 1, 2 \rangle$ and $v = \langle 1, 2, 3 \rangle$ and compute the Klir ([8]) operation $(\frac{uv}{u+v})_E$ compared to the approximation $App = (u * v) : (u + v)$.

Finally, we compare the multiplication $*$ with the arithmetic introduced by Ma et al. in [10]. In our notation, the input fuzzy numbers of their last example are $u =$

$(1, \frac{\alpha(1-\alpha)}{2}, \frac{2-\alpha(\alpha+1)}{2})$ and $v = (2, \frac{\alpha(\alpha-1)}{2}, \frac{2-\alpha(\alpha+1)}{2})$; their multiplication produces the linear symmetric $uv = (2, 0, 1 - \alpha)$ having α -cuts $[uv]_\alpha = [1 + \alpha, 3 - \alpha]$ while $u * v = (2, -(\frac{\alpha(\alpha-1)}{2})^2, (\frac{2-\alpha(\alpha+1)}{2})^2)$.



6. CONCLUSIONS

A decomposition of fuzzy numbers is introduced that isolates three basic components with specific interpretation in the fuzzy context. The decomposition is used to study properties of fuzzy arithmetic, in particular distributivity and symmetry in multiplication and division. We show that distributivity may depend on the form of symmetry of the involved fuzzy quantities and we characterize necessary and sufficient condition for distributivity of multiplication. Further, we use the decomposition to construct possible approximations of fuzzy multiplication and division with the distributivity property. The approximations are compared with other proposed operations in the fuzzy literature and some preliminary results in this direction appear quite interesting for further investigation in modeling fuzzy quantities and understanding properties of fuzzy arithmetic.

7. APPENDIX

We give now some of the (simple but tedious) proofs of the results illustrated in the paper.

7.1. D^* is equivalent to D . We have defined in (14)

$$D^*(u, v) = \widehat{D}(u, v) + \widetilde{D}(u, v) + \overline{D}(u, v).$$

We see that $D \leq D^* \leq 5D$; first, we have $D \leq D^*$ as

$$\begin{aligned} D(u, v) &\leq \max\{|\widehat{u}^- - \widehat{v}^-|, |\widehat{u}^+ - \widehat{v}^+|\} + \sup_{\alpha \in [0,1]} |\widetilde{u}_\alpha - \widetilde{v}_\alpha| + \sup_{\alpha \in [0,1]} |\overline{u}_\alpha - \overline{v}_\alpha| \\ &\leq \widehat{D}(u, v) + \widetilde{D}(u, v) + \overline{D}(u, v) = D^*(u, v). \end{aligned}$$

On the other hand,

$$\widehat{D}(u, v) = \max\{|\widehat{u}^- - \widehat{v}^-|, |\widehat{u}^+ - \widehat{v}^+|\} \leq D(u, v);$$

$$\begin{aligned}
\tilde{D}(u, v) &= \sup_{\alpha} |\tilde{u}_{\alpha} - \tilde{v}_{\alpha}| \\
&= \sup_{\alpha} \left| \left[\frac{1}{2} (u_{\alpha}^{-} + u_{\alpha}^{+}) + \frac{1}{2} (\hat{u}^{-} + \hat{u}^{+}) \right] - \left[\frac{1}{2} (v_{\alpha}^{-} + v_{\alpha}^{+}) + \frac{1}{2} (\hat{v}^{-} + \hat{v}^{+}) \right] \right| \\
&\leq \sup_{\alpha} \left\{ \frac{1}{2} |u_{\alpha}^{-} - v_{\alpha}^{-}| + \frac{1}{2} |u_{\alpha}^{+} - v_{\alpha}^{+}| + \frac{1}{2} |\hat{u}^{-} - \hat{v}^{-}| + \frac{1}{2} |\hat{u}^{+} - \hat{v}^{+}| \right\} \\
&\leq \sup_{\alpha} \{ \max \{ |u_{\alpha}^{-} - v_{\alpha}^{-}|, |u_{\alpha}^{+} - v_{\alpha}^{+}| \} + \max \{ |\hat{u}^{-} - \hat{v}^{-}|, |\hat{u}^{+} - \hat{v}^{+}| \} \} \\
&\leq D(u, v) + \hat{D}(u, v) \leq 2D(u, v);
\end{aligned}$$

$$\begin{aligned}
\overline{D}(u, v) &= \sup_{\alpha} |\overline{u}_{\alpha} - \overline{v}_{\alpha}| \\
&= \sup_{\alpha} \left| \frac{1}{2} (u_{\alpha}^{+} - u_{\alpha}^{-}) - \frac{1}{2} (\hat{u}^{+} - \hat{u}^{-}) - \frac{1}{2} (v_{\alpha}^{+} - v_{\alpha}^{-}) + \frac{1}{2} (\hat{v}^{+} - \hat{v}^{-}) \right| \\
&\leq \sup_{\alpha} \{ \max \{ |u_{\alpha}^{+} - v_{\alpha}^{+}|, |u_{\alpha}^{-} - v_{\alpha}^{-}| \} + \max \{ |\hat{u}^{+} - \hat{v}^{+}|, |\hat{u}^{-} - \hat{v}^{-}| \} \} \\
&\leq \hat{D}(u, v) + D(u, v) \leq 2D(u, v);
\end{aligned}$$

so, $D^{*} = \hat{D} + \tilde{D} + \overline{D} \leq 5D$.

7.2. Computation of α -cuts of uv . In terms of the decomposition, the α -cuts of the fuzzy product uv are given by (20):

$$\begin{aligned}
[uv]_{\alpha} &= [u_{\alpha}^{-}, u_{\alpha}^{+}] [v_{\alpha}^{-}, v_{\alpha}^{+}] \\
&= [\min \{ (\hat{u} + \tilde{u}_{\alpha}) v_{\alpha}^{-} - \overline{u}_{\alpha} |v_{\alpha}^{-}|, (\hat{u} + \tilde{u}_{\alpha}) v_{\alpha}^{+} - \overline{u}_{\alpha} |v_{\alpha}^{+}| \}, \\
&\quad \max \{ (\hat{u} + \tilde{u}_{\alpha}) v_{\alpha}^{-} + \overline{u}_{\alpha} |v_{\alpha}^{-}|, (\hat{u} + \tilde{u}_{\alpha}) v_{\alpha}^{+} + \overline{u}_{\alpha} |v_{\alpha}^{+}| \}].
\end{aligned}$$

In fact,

$$\begin{aligned}
[u_{\alpha}^{-}, u_{\alpha}^{+}] [v_{\alpha}^{-}, v_{\alpha}^{+}] &= [\min \{ u_{\alpha}^{-} v_{\alpha}^{-}, u_{\alpha}^{-} v_{\alpha}^{+}, u_{\alpha}^{+} v_{\alpha}^{-}, u_{\alpha}^{+} v_{\alpha}^{+} \}, \\
&\quad \max \{ u_{\alpha}^{-} v_{\alpha}^{-}, u_{\alpha}^{-} v_{\alpha}^{+}, u_{\alpha}^{+} v_{\alpha}^{-}, u_{\alpha}^{+} v_{\alpha}^{+} \}] \\
&= \left[\min \left\{ \begin{array}{l} (\hat{u} + \tilde{u}_{\alpha}) v_{\alpha}^{-} - \overline{u}_{\alpha} v_{\alpha}^{-}, (\hat{u} + \tilde{u}_{\alpha}) v_{\alpha}^{+} - \overline{u}_{\alpha} v_{\alpha}^{+}, \\ (\hat{u} + \tilde{u}_{\alpha}) v_{\alpha}^{-} + \overline{u}_{\alpha} v_{\alpha}^{-}, (\hat{u} + \tilde{u}_{\alpha}) v_{\alpha}^{+} + \overline{u}_{\alpha} v_{\alpha}^{+} \end{array} \right\}, \right. \\
&\quad \left. \max \left\{ \begin{array}{l} (\hat{u} + \tilde{u}_{\alpha}) v_{\alpha}^{-} - \overline{u}_{\alpha} v_{\alpha}^{-}, (\hat{u} + \tilde{u}_{\alpha}) v_{\alpha}^{+} - \overline{u}_{\alpha} v_{\alpha}^{+}, \\ (\hat{u} + \tilde{u}_{\alpha}) v_{\alpha}^{-} + \overline{u}_{\alpha} v_{\alpha}^{-}, (\hat{u} + \tilde{u}_{\alpha}) v_{\alpha}^{+} + \overline{u}_{\alpha} v_{\alpha}^{+} \end{array} \right\} \right] \\
&= \left[\min \left\{ \begin{array}{l} (\hat{u} + \tilde{u}_{\alpha}) v_{\alpha}^{-} + \min \{ -\overline{u}_{\alpha} v_{\alpha}^{-}, +\overline{u}_{\alpha} v_{\alpha}^{-} \}, \\ (\hat{u} + \tilde{u}_{\alpha}) v_{\alpha}^{+} + \min \{ -\overline{u}_{\alpha} v_{\alpha}^{+}, +\overline{u}_{\alpha} v_{\alpha}^{+} \} \end{array} \right\}, \right. \\
&\quad \left. \max \left\{ \begin{array}{l} (\hat{u} + \tilde{u}_{\alpha}) v_{\alpha}^{-} + \max \{ -\overline{u}_{\alpha} v_{\alpha}^{-}, +\overline{u}_{\alpha} v_{\alpha}^{-} \}, \\ (\hat{u} + \tilde{u}_{\alpha}) v_{\alpha}^{+} + \max \{ -\overline{u}_{\alpha} v_{\alpha}^{+}, +\overline{u}_{\alpha} v_{\alpha}^{+} \} \end{array} \right\} \right];
\end{aligned}$$

But $\overline{u}_{\alpha} \geq 0$ and then $\min \{ -\overline{u}_{\alpha} v_{\alpha}^{-}, +\overline{u}_{\alpha} v_{\alpha}^{-} \} = -\overline{u}_{\alpha} |v_{\alpha}^{-}|$, $\min \{ -\overline{u}_{\alpha} v_{\alpha}^{+}, +\overline{u}_{\alpha} v_{\alpha}^{+} \} = -\overline{u}_{\alpha} |v_{\alpha}^{+}|$, $\max \{ -\overline{u}_{\alpha} v_{\alpha}^{-}, +\overline{u}_{\alpha} v_{\alpha}^{-} \} = \overline{u}_{\alpha} |v_{\alpha}^{-}|$ and $\max \{ -\overline{u}_{\alpha} v_{\alpha}^{+}, +\overline{u}_{\alpha} v_{\alpha}^{+} \} = \overline{u}_{\alpha} |v_{\alpha}^{+}|$.

7.3. Proof of distributivity conditions. We have distributivity if and only if the minimum (and resp. the maximum) in $[(u + v)z]_{\alpha}$ is equal to the sums of the two minima (and resp. the two maxima) in $[uz + vz]_{\alpha}$.

To simplify the notation we pose

$$\begin{aligned} a_\alpha &= \widehat{u} + \widetilde{u}_\alpha, & a'_\alpha &= \widehat{v} + \widetilde{v}_\alpha, \\ b_\alpha &= \overline{u}_\alpha \geq 0, & b'_\alpha &= \overline{v}_\alpha \geq 0, \\ c_\alpha &= z_\alpha^-, & d_\alpha &= z_\alpha^+ \end{aligned}$$

and define

$$\begin{aligned} m_\alpha &= \min\{(a_\alpha + a'_\alpha)c_\alpha - (b_\alpha + b'_\alpha)|c_\alpha|, (a_\alpha + a'_\alpha)d_\alpha - (b_\alpha + b'_\alpha)|d_\alpha|\} \\ n_\alpha &= \min\{a_\alpha c_\alpha - b_\alpha|c_\alpha|, a_\alpha d_\alpha - b_\alpha|d_\alpha|\} \\ n'_\alpha &= \min\{a'_\alpha c_\alpha - b'_\alpha|c_\alpha|, a'_\alpha d_\alpha - b'_\alpha|d_\alpha|\} \\ M_\alpha &= \max\{(a_\alpha + a'_\alpha)c_\alpha + (b_\alpha + b'_\alpha)|c_\alpha|, (a_\alpha + a'_\alpha)d_\alpha + (b_\alpha + b'_\alpha)|d_\alpha|\} \\ N_\alpha &= \max\{a_\alpha c_\alpha + b_\alpha|c_\alpha|, a_\alpha d_\alpha + b_\alpha|d_\alpha|\} \\ N'_\alpha &= \max\{a'_\alpha c_\alpha + b'_\alpha|c_\alpha|, a'_\alpha d_\alpha + b'_\alpha|d_\alpha|\} \end{aligned}$$

so that

$$[(u + v)z]_\alpha = [m_\alpha, M_\alpha] \text{ and } [uz + vz]_\alpha = [n_\alpha + n'_\alpha, N_\alpha + N'_\alpha].$$

We then have to verify in which cases $m_\alpha = n_\alpha + n'_\alpha$ and $M_\alpha = N_\alpha + N'_\alpha$. For the minimum, we have the two cases (we omit the subscript α)

$$\begin{aligned} \text{(i)} \quad & \begin{cases} n = ac - b|c|, & n' = a'c - b'|c| \\ \downarrow \\ m = (a + a')c - (b + b')|c| = n + n' \end{cases} \\ \text{(i)'} \quad & \begin{cases} n = ad - b|d|, & n' = a'd - b'|d| \\ \downarrow \\ m = (a + a')d - (b + b')|d| = n + n' \end{cases} \end{aligned}$$

and, for the maximum, the two cases

$$\begin{aligned} \text{(ii)} \quad & \begin{cases} N = ac + b|c|, & N' = a'c + b'|c| \\ \downarrow \\ M = (a + a')c + (b + b')|c| = N + N' \end{cases} \\ \text{(ii)'} \quad & \begin{cases} N = ad + b|d|, & N' = a'd + b'|d| \\ \downarrow \\ M = (a + a')d + (b + b')|d| = N + N'. \end{cases} \end{aligned}$$

Combining (i) or (i)' with (ii) or (ii)', gives the four systems of inequalities:

$$\begin{aligned} & \begin{cases} ac - b|c| \leq ad - b|d| \\ a'c - b'|c| \leq a'd - b'|d| \\ ad + b|d| \leq ac + b|c| \\ a'd + b'|d| \leq a'c + b'|c| \end{cases} \quad \text{(i) and (ii)} \\ & \begin{cases} ac - b|c| \leq ad - b|d| \\ a'c - b'|c| \leq a'd - b'|d| \\ ac + b|c| \leq ad + b|d| \\ a'c + b'|c| \leq a'd + b'|d| \end{cases} \quad \text{(i) and (ii)'} \end{aligned}$$

$$\left\{ \begin{array}{l} ad - b|d| \leq ac - b|c| \\ a'd - b'|d| \leq a'c - b'|c| \\ ad + b|d| \leq ac + b|c| \\ a'd + b'|d| \leq a'c + b'|c| \end{array} \right. \quad \text{(i)' and (ii)}$$

$$\left\{ \begin{array}{l} ad - b|d| \leq ac - b|c| \\ a'd - b'|d| \leq a'c - b'|c| \\ ac + b|c| \leq ad + b|d| \\ a'c + b'|c| \leq a'd + b'|d|. \end{array} \right. \quad \text{(i)' and (ii)'}$$

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